Critical behavior of nonequilibrium phase transitions to magnetically ordered states

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We describe nonequilibrium phase transitions in arrays of dynamical systems with cubic nonlinearity driven by multiplicative Gaussian white noise. Depending on the sign of the spatial coupling we observe transitions to ferromagnetic or antiferromagnetic ordered states. We discuss the phase diagram, the order of the transitions, and the critical behavior. For global coupling we show analytically that the critical exponent of the magnetization exhibits a transition from the value 1/2 to a nonuniversal behavior depending on the ratio of noise strength to the magnitude of the spatial coupling.

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I. INTRODUCTION

In the last decade studying arrays of stochastically driven nonlinear dynamical systems the notion of noise induced nonequilibrium phase transition has been established [1–10]; for a recent monograph see Ref. [11]. In close analogy to equilibrium phase transition one has order parameters and finds continuous or discontinuous transitions associated with ergodicity breaking. The behavior near the transition point is characterized by power laws and a critical slowing down.

In this paper we consider arrays of spatially harmonically coupled Stratonovich models [12] that undergo transitions into ordered states comparable to ferromagnetic (FM) or antiferromagnetic (AFM) phases depending on the sign of the coupling constant. The AFM situation is described first in this paper for that class of models. We determine the phase diagram and characterize the critical behavior at these transitions. For the globally coupled system we derive an analytical result for the critical exponent of the order parameter, i.e., the magnetization. This critical exponent exhibits a hitherto not described transition from a value 1/2 to a nonuniversal behavior when increasing the ratio of noise strength and magnitude of the spatial coupling.

The dynamics of the individual constituents x_i at the lattice sites $i=1, \ldots ,L$ is governed by a system of stochastic ordinary differential equations in the Stratonovich sense

$$\dot{x}_i = a x_i - x_i^3 + x_i \xi_i - \frac{D}{N} \sum_{j \in \mathcal{N}(i)} (x_i - x_j), \tag{1}$$

where $\mathcal{N}(i)$ denotes the set of sites interacting with site i. N, the number of sites belonging to this set is equal to $L-1$ in the case of global coupling and to $2d$ in the case of nearest neighbor (NN) coupling on a simple cubic lattice in d dimensions. D is the strength of the spatial interactions. $\xi_i(t)$ is a zero mean spatially uncorrelated Gaussian white noise with autocorrelation function $\langle \xi_i(t) \xi_j(t') \rangle = \sigma^2 \delta_{ij} \delta(t-t')$, where σ^2 is the noise strength.

The stationary probability density $P_s(x_i)$ fulfills the (reduced) stationary Fokker-Planck equation [2]

$$0 = \frac{\partial}{\partial x_i} \left[\left(-a x_i + x_i^3 + \frac{D}{N} \sum_{j \in \mathcal{N}(i)} (x_i - \langle x_j | x_i \rangle) + \frac{\sigma^2}{2} x_i \frac{\partial}{\partial x_i} x_i \right) P_s(x_i) \right], \tag{2}$$

where $\langle x_j | x_i \rangle = \int dx_j x_j P_s(x_j | x_i)$ is the steady state conditional average of x_j, $j \in \mathcal{N}(i)$, given x_i at site i. We denote its spatial average by

$$m_i = \frac{1}{N} \sum_{j \in \mathcal{N}(i)} \langle x_j | x_i \rangle. \tag{3}$$

II. GLOBAL COUPLING

In the case of global coupling, fluctuations of m_i disappear in the limit $L \to \infty$. We thus may consider m_i as a parameter and obtain except for a constant factor a stationary solution of Eq. (2)

$$p_s(x_i, m_i) = |x_i|^{2(a-D)/\sigma^2 - 1} e^{-(x_i^2 + 2D m_i / x_i)/\sigma^2}. \tag{4}$$

If this expression is normalizable, the stationary probability density $P_s(x_i, m_i)$ reads

$$P_s(x_i, m_i) = \begin{cases} [1/N(m_i)] p_s(x_i, m_i) & \text{for } x_i \in \text{support}, \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

where $N(m_i) = \int_{\text{supp}} dx p_s(x, m_i)$. P_s lives on a support on which Eq. (4) is normalizable, i.e., N is finite.

For both D and m_i nonzero the support of P_s is such that $D m_i / x_i \geqslant 0$ ensuring normalizability of Eq. (4). For $m_i = 0$ normalizability requires that the exponent of the algebraic factor in Eq. (4) is larger than -1, i.e., $D < a$. For $D > a$ the solution (4) is not normalizable and we have $P_s(x_i) = \delta(x_i)$. The determination of m_i is described below in detail.

Varying the control parameters of the system a and D, or the strength of the noise σ^2, one obtains the phase diagram shown in Fig. 1.

We first consider $D > 0$ which favors a FM order. In the spatially homogeneous case $m_i \equiv m$ and for $m > 0$ or $m < 0$ the support of P_s is $[0, \infty)$ and $(-\infty, 0]$, respectively.

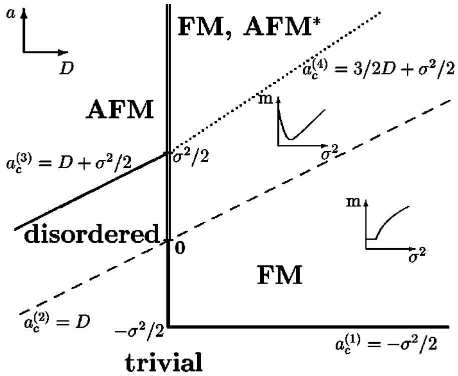

FIG. 1. Phase diagram in the (D,a) plane for global coupling. Continuous transitions toward FM or AFM states occur crossing the thick solid lines. The double solid line indicates discontinuous transitions. The critical values $a_c^{(n)}, n=1,\ldots,4$, the disordered phase, the metastable AFM* phase, and the insets are explained in the text.

All constituents have the same (statistical or temporal) average

$$\langle x\rangle = \int_{\mathrm{supp}} dx\, x P_s(x,m) = \lim_{T\to\infty}\frac{1}{T}\int_0^T dt\, x(t) = F(m), \quad (6)$$

which equals the spatial average m (ergodicity). This leads to the self-consistency condition

$$m = \langle x\rangle = F(m) \quad (7)$$

determining m. One easily finds

$$F(\pm 0) = \begin{cases} \pm\sigma\Gamma\left(\dfrac{a-D}{\sigma^2}+\dfrac{1}{2}\right)\Gamma\left(\dfrac{a-D}{\sigma^2}\right) & \text{for } a>D,\\ 0 & \text{for } a<D. \end{cases} \quad (8)$$

For $a<a_c^{(1)}=-\sigma^2/2$ Eq. (7) has a trivial stable solution $m=0$ that loses its stability at $a=a_c^{(1)}$, which is determined by the condition $F'(0)=1$. It bifurcates into a pair of stable solutions $m=m_+>0$ and $m=-m_+=m_-$ corresponding to a continuous transition from a paramagnetic to a FM situation. Choosing $m=m_+$, for instance, the stationary probability distribution of the corresponding ergodic component is $P_s(x,m_+)$, cf. Eq. (5). In the FM region, for $a_c^{(1)}<a<a_c^{(2)}=D$ the magnetization $m=\langle x\rangle$ increases monotonously with σ^2, whereas for $a>a_c^{(2)}$ there is a nonmonotonous behavior, cf. Fig. 2.

As a function of D, the magnetization m increases continuously from zero when increasing D from zero for $a_c^{(1)}<a<0$, whereas the transition is discontinuous for $a>0$ as shown in Fig. 3, cf. also Ref. [8].

Within the FM region, a metastable [13] antiferromagnetic solution (AFM*) exists besides the stable FM solution for $a>a_c^{(4)}$, see Fig. 1. The critical value $a_c^{(4)}=3/2D+\sigma^2/2$ is obtained for weak noise from an extremal approximation for $m=\langle x\rangle$ in the spirit of Ref. [5].

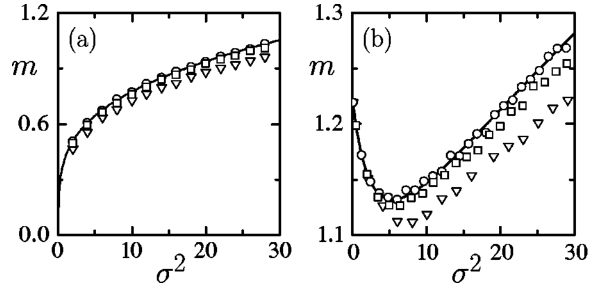

FIG. 2. Order parameter m vs σ^2. The solid line is the solution of Eq. (7) compared with simulations of Eq. (1) for global (circles) and NN coupling in $d=1$ (triangles) and $d=3$ (squares). For $a<a_c^{(2)}=D$ (a) the order parameter m increases monotonously with σ^2 whereas for $a>D$ (b) a pronounced minimum appears. The parameters used are $a=0$ in (a) and $a=1.5$ in (b), $D=0.5$.

For $D<0$ the situation is different. For $a<a_c^{(2)}$ we have $P_s(x)=\delta(x)$. In the range $a_c^{(2)}<a<a_c^{(3)}=D+\sigma^2/2$ one finds $m=0$ and the stationary probability density $P_s(x)$ lives on $(-\infty,\infty)$, we call this the disordered phase. For $a>a_c^{(3)}$ the stationary solution (4) is normalizable only for $m\neq 0$, for $m>0$ or $m<0$ the support is $(-\infty,0]$ or $[0,\infty)$, respectively. We define two subsystems labeled by $+$ and $-$ for which the averages $\langle x_i\rangle$ have $+$ or $-$ sign, respectively. For global coupling AFM order implies $m_\pm\to\mp 0$ in the limit $L\to\infty$. Therefore, the mean values $\langle x_i\rangle=\langle x_\pm\rangle$ are given by

$$\langle x_\pm\rangle = -\langle x_\mp\rangle = \int_0^{\pm\infty} dx\, x P_s(x,\mp 0) = \pm F(\pm 0), \quad (9)$$

where P_s is taken from Eq. (5).

III. NEAREST NEIGHBOR COUPLING

For NN coupling on a cubic lattice a mean field approximation is obtained in a similar way replacing the spatial average over the $2d$ nearest neighbors as $m_i=1/(2d)\Sigma_{j\in\mathcal{N}(i)}\langle x_j|x_i\rangle\approx\langle x_i\rangle$. The FM case, $D>0$, is formally the same as for global coupling but Eq. (7) holds only approximately. In the AFM case, $D<0$, one should take into account that now the two subsystems $+$ and $-$ correspond to different Néel sublattices A and B, respectively, and all the nearest neighbors of a given lattice site belong to the

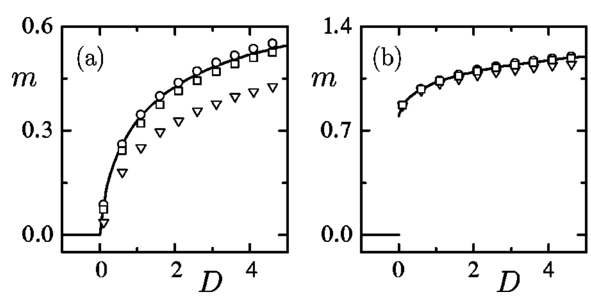

FIG. 3. Order parameter m vs D. Lines and symbols have the same meaning as in Fig. 2. For $-\sigma^2/2=a_c^{(1)}<a<0$ the transition is continuous (a), whereas for $0<a$ it is discontinuous (b). The parameters used are $a=-0.5$ in (a) and $a=1$ in (b), $\sigma^2=2$.

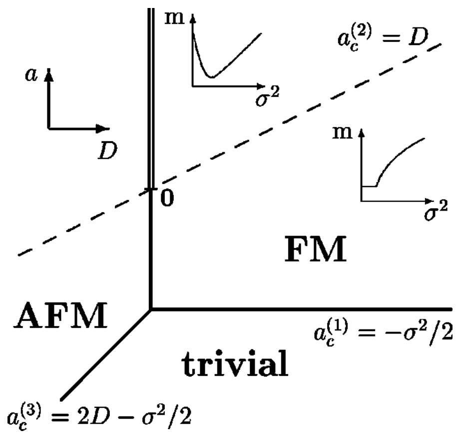

FIG. 4. Phase diagram in the (D,a) plane for NN coupling. For $D>0$ the situation is the same as for global coupling except that the metastable AFM* phase is absent here. For the case $D<0$ see text.

complementary sublattice. Self-consistency requires

$$m_{\pm} = -\langle x_{\pm}\rangle = -\int_0^{\pm\infty} dx\, x P_s(x,m_{\pm}) = -m_{\mp}. \quad (10)$$

For $D<0$ system (1) is invariant under the transformation [15] $x_i \to -x_i$ for $i\in A$, $x_j \to x_j$ for $j \in B$, $D\to -D$, $a\to a-2D$. This implies that properties of the AFM phase for spatial coupling $D=-D'<0$ can be inferred from properties of the FM phase for spatial coupling strength D', cf. Ref. [16]. For instance, $a_c^{(1)} = -\sigma^2/2$ transforms into $a_c^{(3)} = 2D - \sigma^2/2$. The phase diagram for NN coupling is shown in Fig. 4.

IV. CRITICAL BEHAVIOR

Varying the control parameters a and σ^2 one observes continuous transitions from zero to nonzero values of m with a characteristic power law behavior near the critical values of the control parameters. To analyze the critical behavior it is useful to write the self-consistency equations (7) and (10) in compact form as

$$m = -\frac{2|D|}{\sigma^2}\left(\frac{\partial \ln I(m)}{\partial m}\right)^{-1}, \quad (11)$$

where

$$I(m) = \int_0^\infty dx\, x^{2(a-D)/\sigma^2} e^{-(x^2+2|D|m/x)/\sigma^2}. \quad (12)$$

In the limits $\sigma\to 0$ or $D\to\infty$ this integral can be evaluated by the Laplace method, cf., e.g., Ref. [17]. Inserting the results in Eq. (11) for small m, one obtains the power laws $m\sim(a+\sigma^2/2+|D|-D)^{1/2}$ for $\sigma\to 0$ and $m\sim(a+\sigma^2/2)^{1/2}$ for $D\to\infty$ with the critical exponent $\beta=1/2$, cf. also Refs. [2,4,8].

For finite values of σ and D the scaling behavior of $I(m)$ can be evaluated for small m with the result [18]

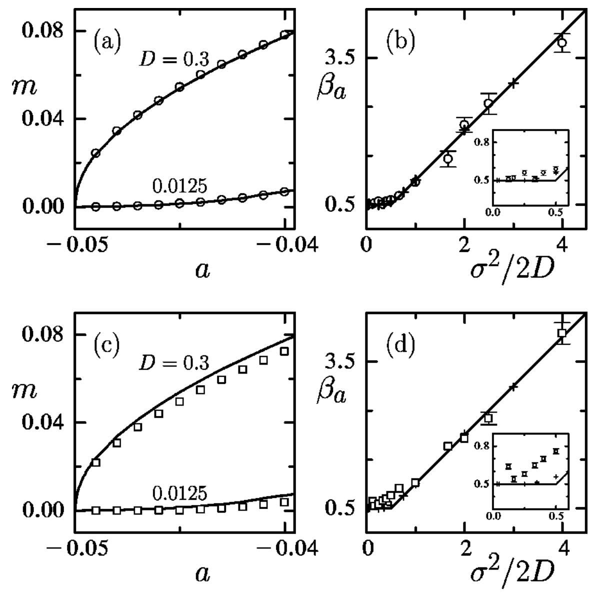

FIG. 5. Critical behavior. The figure shows the order parameter m vs control parameter a for different values of D ($\sigma^2=0.1$, i.e., $a_c=-0.05$) and the critical exponent β_a vs $\sigma^2/(2D)$. (a) and (b) refer to global coupling, (c) and (d) to NN coupling in $d=3$, respectively. In (a) and (c) the solid line is the numerical solution of the self-consistency equation (11). In (b) and (d) the solid line is the analytical result (14); the + symbols represent the numerical solution of Eq. (11). Circles and squares result from simulations of Eq. (1). Error bars are partially smaller than the symbol size.

$$I(m)\sim m^{2(\varepsilon-D)/\sigma^2}(1+C_1 m^{-2(\varepsilon-D)/\sigma^2}+C_2 m^2), \quad (13)$$

where $\varepsilon=a-a_c$. The critical value a_c is $-\sigma^2/2$ for the FM case and $2D-\sigma^2/2$ for the AFM case. Inserting Eq. (13) in Eq. (11) we obtain for small m and in lowest order of ε the power law

$$m\sim\varepsilon^{\beta}, \quad \beta=\sup\{1/2,\sigma^2/(2|D|)\}, \quad (14)$$

logarithmic corrections are easily computed. Obviously, the critical exponents are the same varying a or σ^2, i.e., $\beta_a=\beta_\sigma=\beta$ using notations from Ref. [8]. For models where the cubic nonlinearity in Eq. (1) is replaced by x^{p+1}, $p>0$, in Eq. (14) the value 1/2 is replaced by $1/p$ [19].

Figure 5 compares the magnetization $m(a)$ and the critical exponent β obtained from the analytical results with simulations for both global and NN coupling [20]. For global coupling, simulations for systems of size 10^3 are already very close to the results for the infinite system.

V. CONCLUSION

For the globally coupled model we found analytically a transition of the critical exponent β from a value 1/2, which reflects the order of the nonlinearity and is independent of the strength of noise σ^2 and spatial coupling D, to a nonuniversal behavior, depending on σ^2 and D independent of the order of the nonlinearity. This differs from the value $\beta=1$ proposed for the continuous version of the model in Ref. [8].

Critical exponents that depend continuously on parameters (continuous exponents) have been found earlier in different context both in equilibrium (e.g., Refs. [21,22]) and nonequilibrium statistical mechanics (e.g., Refs. [23,24]). For a model similar to ours, Giada and Marsili [24] obtained asymptotically for large values of the critical exponent β the result $\beta=\sigma^2/2$ which is compatible with our more general result (14) putting the spatial coupling strength $D=1$.

If the noise is not too strong, the "mean field" results describe the critical behavior for NN coupling observed in our simulations very well, cf. Fig. 5(d). Also the numerical result $\beta\approx 1$ obtained by Genovese and Muñoz [8] near $a=-1$, $\sigma^2=2$ for $D=1$ (their "weak noise phase") is in accord with our analytical result (14). For stronger noise, simulations for NN coupling may differ considerably from the mean-field prediction.

For models with a nonlinearity x^{p+1}, the value $1/p$ of β is in general different from the value 1/2 characteristic for models with only additive noise. In this case, one has to expect interesting crossover phenomena for models with additive and multiplicative noise when changing the relative strength of the noises.

ACKNOWLEDGMENTS

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[13] We prepared the system such that a fraction λ of the constituents have positive initial values and a fraction $1-\lambda$ negative ones. By simulation we determined the first passage times for which one of the constituents changes the sign. For $\lambda\neq 1/2$, the mean first passage time (MFPT) decreases exponentially with the system size L and the system reaches the FM state very fast. For $\lambda=1/2$ the MFPT increases with L and is expected to diverge for $L\to\infty$. Additive noise, however, naturally leads to $\lambda\neq 1/2$ and finally to a FM state; the AFM* state is therefore only metastable, cf. also Ref. [14].

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[18] Substituting $y=x/m$ in Eq. (12) leads to $I(m)=m^{1+2(a-D)/\sigma^2}J(m)$. We split $J=\int_0^{y'}\cdots+\int_{y'}^{\infty}\cdots=J_1+J_2$ at an intermediate point y' with $2|D|/\sigma^2\ll y'\ll\sigma/m$. In J_1 and J_2 we expand $\exp\{-m^2y^2/\sigma^2\}$ and $\exp\{-2|D|/(\sigma^2 y)\}$ to second order, respectively. One readily obtains $J_1\sim c_1+c_2m^2$. To treat J_2 further we resubstitute $x=ym$ obtaining $J_2=m^{-1-2(a-D)/\sigma^2}I_2$. We split now $I_2=\int_{my'}^{x'}\cdots+\int_{x'}^{\infty}\cdots=I_2^a+I_2^b$ at a second intermediate point x' with $my'<x'\ll\sigma$. In I_2^a we expand $\exp\{-x^2/\sigma^2\}$ to second order and obtain $I_2^a\sim c_3+c_4m^{1+2(a-D)/\sigma^2}+c_5m^{3+2(a-D)/\sigma^2}$; $I_2^b=c_6$. All c_n, $n=1,\ldots,6$, are independent of m. Introducing $\varepsilon=a-a_c$ and renaming the constants gives Eq. (13).

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[20] System (1) was solved by a fourth-order Runge-Kutta scheme adopted to the stochastic case such that the noise is set constant during one time step. The Gaussian white noise was generated by a Box-Müller algorithm. After thermalization we computed m as the temporal average over 5×10^4 time steps of the spatial mean of x_i over all lattice sites. The time steps were chosen small enough to avoid numerical instabilities ($\Delta t=10^{-3}\ldots 10^{-2}$). To determine a_c numerically, we exploited that for $a<a_c$ starting with initial conditions corresponding to $m>0$ the system relaxes in a finite time to $m'<m$. m' serves as initial condition for $a'=a+\delta a$. Iterating the procedure, m relaxes further as long as we are below a_c, above a_c it increases. The critical exponent β was obtained from the slope of a linear fit of the plot $\ln m$ vs $\ln(a-a_c)$ in the range where the first derivative of $\ln m$ is almost constant. The parameter

dependence of m was computed for systems of size $L=10\times10\times10$. Critical exponents were obtained considering larger systems, $L=40\times40\times40$. Periodic boundary conditions were used for the cubic lattice.

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