ATS 602 – Homework 4

due Friday, April 7

1. **Two-layer baroclinic instability**

   (a) Consider the dispersion relation:

   $$\hat{c}_{1,2} = -\frac{1}{\hat{K}^2}\frac{\hat{K}^2+1}{\hat{K}^2+2}\left[1 \pm \frac{\sqrt{4+\hat{K}^4(\hat{K}^4-4)(\Delta\hat{U})^2}}{2(\hat{K}^2+1)}\right],$$

   with $\hat{c} \equiv (k_d^2/\beta)c$, $\Delta\hat{U} \equiv (k_d^2/\beta)\Delta U$, and $\hat{K} \equiv K/k_d$.
   Derive an analytical expression for the maximum growth rate and the associated wave number of unstable modes (you may set $l = 0$ for this). (Feel free to illustrate things graphically, similar to the plots provided in the notes.) Confirm the special case of $\beta = 0$ as given on page 15 of the notes. Physically discuss the roles that β, the basic state shear, and the basic state stratification play in the stability of the flow.

   (b) Derive an expression for the meridional group velocity of unstable modes ($c_{g,y} = \partial_l \omega_r$, where $\omega_r = c_r k$ is the real part of the frequency). Specifically, show that $c_{g,y}$ shares the usual Rossby wave property that its sign is determined by the sign of kl.

   (c) Derive an expression for the momentum flux of unstable modes, $\overline{v'u'} = -\overline{\partial_x\psi'\partial_y\psi'}$ (the bar represents a spatial average). Specifically, show that $\overline{v'u'} \sim -kle^{2\sigma t}$. Interpret physically what it means that this momentum flux has the opposite sign of $c_{g,y}$. How does this flux depend on the growth rate (e.g. does it become zero for stable/neutral modes)? Also show that the momentum flux is the same for both layers, except for a constant factor.

   (d) Qualitatively discuss the phase difference between the upper and lower layer waves (in terms of their streamfunction) for i) stable/neutral modes, ii) growing modes, iii) decaying modes? Approach this from the perspective of the associated heat fluxes (recall the sign of the heat flux for each case).

   (e) *Optional.* Derive an expression for the ratio of perturbation potential to kinetic energy $\mathcal{P}/\mathcal{K}$ and discuss for the three cases under (d).